13) Appendix II: Scalar wave equation in spherical coordinates

1. Separation of variables

$\nabla^2\psi + k^2\psi = 0$ in spherical coordinates $\Rightarrow$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} + k^2 f = 0$$

The equation is separable $\Rightarrow$ look for $\psi = f_1(r) f_2(\theta) f_3(\varphi)$

$$\frac{f_2 f_3}{r^2}\partial_r\left(r^2\frac{\partial f_1}{\partial r}\right) + \frac{f_1 f_3}{r^2 s\theta}\partial_\theta\left(s\theta\,\partial_\theta f_2\right) + \frac{f_1 f_2}{r^2 s^2\theta}\partial^2_{\varphi^2} f_3 + k^2 f_1 f_2 f_3 = 0$$

Divide by $\psi = f_1 f_2 f_3 \neq 0$:

$$\frac{1}{f_1 r^2}\partial_r\left(r^2\partial_r f_1\right) + \frac{1}{r^2 f_2 s\theta}\partial_\theta\left(s\theta\,\partial_\theta f_2\right) + \frac{1}{f_3 r^2 s^2\theta}\partial^2_{\varphi^2} f_3 + k^2 = 0$$

Multiply by r^2:

$$\frac{1}{f_1}\partial_r\left(r^2\partial_r f_1\right) + k^2 r^2 = -\left[\frac{1}{f_2 s\theta}\partial_\theta\left(s\theta\,\partial_\theta f_2\right) + \frac{1}{f_3 s^2\theta}\partial^2_{\varphi^2} f_3\right] = p^2$$

where $p \in \mathbb{C}$ is our first separation constant. We get:

$$\partial_r\left(r^2\partial_r f_1\right) + \left(k^2 r^2 - p^2\right) f_1 = 0$$

$$p^2 s^2\theta + \frac{s\theta}{f_2}\partial_\theta\left(s\theta\,\partial_\theta f_2\right) = \frac{-1}{f_3}\partial^2_{\varphi^2} f_3 = q^2 \ , \ q \in \mathbb{C}$$

$$\Rightarrow \boxed{\begin{aligned} & r^2\partial^2_{r^2} f_1 + 2r\,\partial_r f_1 + \left(k^2 r^2 - p^2\right) f_1 = 0 \quad (1) \\ & \frac{1}{s\theta}\partial_\theta\left(s\theta\,\partial_\theta f_2\right) + \left(p^2 - \frac{q^2}{s^2\theta}\right) f_2 = 0 \quad (2) \\ & \partial^2_{\varphi^2} f_3 + q^2 f_3 = 0 \quad (3) \end{aligned}}$$

2. Solving the angular part

p and q are separation constants whose choice is governed by the fact that the field must be single valued. In particular, $f_3(\varphi)$ must have period $2\pi \Rightarrow$ $q = 0, \pm 1, \pm 2, \dots, \pm m$ and f_3 is generated by $\cos m\varphi$ and $\sin m\varphi$, $m \in \mathbb{Z}$.

$$\boxed{f_3 = \begin{cases} \cos m\varphi \\ \sin m\varphi \end{cases}}$$

To determine p, we make the variable change $\eta = \cos\theta$ in (2), i.e

$$\begin{cases} \dfrac{df_2}{d\theta} = \dfrac{df_2}{d\eta} \times \underbrace{\dfrac{d\eta}{d\theta}}_{} = -s\theta \dfrac{df_2}{d\eta} = -\sqrt{1-\eta^2}\,\dfrac{df_2}{d\eta} \\ \dfrac{d^2f_2}{d\theta^2} = \dfrac{d}{d\theta}\left(\dfrac{df_2}{d\theta}\right) = \dfrac{d}{d\eta}\left(\dfrac{df_2}{d\theta}\right) \times \dfrac{d\eta}{d\theta} = -s\theta\left(-\sqrt{1-\eta^2}\,\dfrac{d^2f_2}{d\eta} + \dfrac{\cancel{2}\eta}{\cancel{2}\sqrt{1-\eta^2}}\,\dfrac{df_2}{d\eta}\right) \\ \dfrac{d^2f_2}{d\theta^2} = (1-\eta^2)\dfrac{d^2f_2}{d\eta} - \eta\dfrac{df_2}{d\eta} \end{cases}$$

$$\overset{(2)}{\Rightarrow} \frac{1}{s\theta} c\theta \frac{df_2}{d\theta} + \frac{d^2f_2}{d\theta^2} + \left(p^2 - \frac{q^2}{s^2\theta}\right) f_2 = 0 \quad \text{becomes}$$

$$\frac{\overset{\eta}{c\theta}}{\cancel{s\theta}}\left(-\cancel{s\theta}\,\frac{df_2}{d\eta}\right) + (1-\eta^2)\frac{d^2f_2}{d\eta} - \eta\frac{df_2}{d\eta} + \left(p^2 - \frac{q^2}{1-\eta^2}\right) f_2 = 0$$

$$\Rightarrow \boxed{(1-\eta^2)\frac{d^2f_2}{d\eta} - 2\eta\frac{df_2}{d\eta} + \left(p^2 - \frac{m^2}{1-\eta^2}\right) f_2 = 0} \quad (4)$$

Let's focus first on the $m = 0$ case. We get Legendre equation:

$$(1-\eta^2)\frac{d^2f_2}{d\eta} - 2\eta\frac{df_2}{d\eta} + p^2 f_2 = 0$$

It is known (maths) that the only non-singular solutions at $\eta = \pm 1$, are for $p^2 = n(n+1)$ and are the Legendre polynomials $v = P_n(\eta)$, solution of:

$$(1-\eta^2)\frac{d^2v}{d\eta^2} - 2\eta\frac{dv}{d\eta} + n(n+1)v = 0 \quad (\mathcal{L})$$

To solve the general $m \neq 0$ case, we differentiate m times the Legendre equation with respect to η, and get, writing $w = d^m v / d\eta^m$

$$(1-\eta^2)\frac{d^2 w}{d\eta^2} - 2(m+1)\eta\frac{dw}{d\eta} + \left[n(n+1) - m(m+1)\right] w = 0 \qquad (5)$$

(proof is straightforward by recurrence, exercise)

Now writing $w = (1-\eta^2)^{-\frac{m}{2}} f_2(\eta)$, we have

$$\frac{dw}{d\eta} = \left(\cancel{-}\frac{m}{\cancel{2}}\right)\left(\cancel{-}\cancel{2}\eta\right)(1-\eta^2)^{-\frac{m}{2}-1} f_2 + (1-\eta^2)^{-\frac{m}{2}}\frac{df_2}{d\eta}$$

$$= m\eta(1-\eta^2)^{-\frac{m}{2}-1} f_2(\eta) + (1-\eta^2)^{-m/2}\frac{df_2}{d\eta}$$

$$\frac{d^2 w}{d\eta^2} = \left[m(1-\eta^2)^{-\frac{m}{2}-1} + m\eta\left(-\frac{m}{2}-1\right)(-2\eta)(1-\eta^2)^{-\frac{m}{2}-2}\right] f_2(\eta)$$
$$+ 2m\eta(1-\eta^2)^{-\frac{m}{2}-1}\frac{df_2}{d\eta} + (1-\eta^2)^{-m/2}\frac{d^2 f_2}{d\eta^2}$$

$$\Rightarrow -2(m+1)\eta\frac{dw}{d\eta} = (1-\eta^2)^{-m/2}\left(-2(m+1)\eta\frac{df_2}{d\eta} - 2m(m+1)\eta^2(1-\eta^2)^{-1} f_2\right)$$

$$\Rightarrow (1-\eta^2)\frac{d^2 w}{d\eta^2} = (1-\eta^2)^{-m/2}\left\{(1-\eta^2)\frac{d^2 f_2}{d\eta^2} + 2m\eta\frac{df_2}{d\eta} + \left[m + 2m\left(\frac{m}{2}+1\right)\eta^2(1-\eta^2)^{-1}\right] f_2\right\}$$

$$\Rightarrow (1-\eta^2)\frac{d^2 w}{d\eta^2} - 2(m+1)\eta\frac{dw}{d\eta} + \left[n(n+1) - m(m+1)\right] w = 0$$

$$\Rightarrow (1-\eta^2)^{-m/2}\left\{(1-\eta^2)\frac{d^2 f_2}{d\eta^2} - 2\eta\frac{df_2}{d\eta} + \left(\cancel{m} - m^2\eta^2(1-\eta^2)^{-1} + n(n+1) - m^2 - \cancel{m}\right) f_2\right\} = 0$$

$$\Rightarrow (1-\eta^2)\frac{d^2 f_2}{d\eta^2} - 2\eta\frac{df_2}{d\eta} + \left[n(n+1) - m^2\left(1 + \frac{\eta^2}{1-\eta^2}\right)\right] f_2 = 0$$

$$\Rightarrow (1-\eta^2)\frac{d^2 f_2}{d\eta^2} - 2\eta\frac{df_2}{d\eta} + \left[n(n+1) - \frac{m^2}{1-\eta^2}\right] = 0$$

(4) is satisfied!

3. Surface spherical harmonics

v satisfies $(\mathcal{L}) \Leftrightarrow f_2 = (1-\eta^2)^{m/2} \dfrac{d^m v}{d\eta^m}$ satisfies (4)

Legendre polynomials $\Leftrightarrow$ Associated Legendre polynomials

$v = P_n(\eta)$ $\qquad$ $f_2 = P_n^m(\eta) = (1-\eta^2)^{m/2} \dfrac{d^m v}{d\eta^m}$

Rodrigues formula: $P_n(\eta) = \dfrac{1}{2^n n!} \dfrac{d^n}{d\eta^n}\left[(\eta^2-1)^n\right]$

$\Rightarrow$ P_n is a polynomial of order n $\quad$ demonstration may be omitted in class

The solution of (4) is generated by the associated Legendre polynomials

$$\Rightarrow \boxed{f_2(\eta) = P_n^m(\eta) = \frac{(1-\eta^2)^{m/2}}{2^n n!} \frac{d^{n+m}\left[(\eta^2-1)^n\right]}{d\eta^{n+m}}} \quad (6) \qquad \eta = \cos\theta$$

also called zonal harmonics

* Rq: The definition of P_n^m above holds only when $m, n \in \mathbb{N}$.
With such convention, $P_n^m = 0$ when $m > n$.

* Examples:

$P_0(z) = 1$

$P_1(z) = z = \cos\theta$

$P_1^1(z) = (1-z^2)^{1/2} = \sin\theta$

$P_2(z) = \frac{1}{2}(3z^2-1) = \frac{1}{4}(3\cos 2\theta - 1)$

$P_2^1(z) = 3(1-z^2)^{1/2} z = \frac{3}{2}\sin 2\theta$

$P_2^2(z) = 3(1-z^2) = \frac{3}{2}(1-\cos 2\theta)$

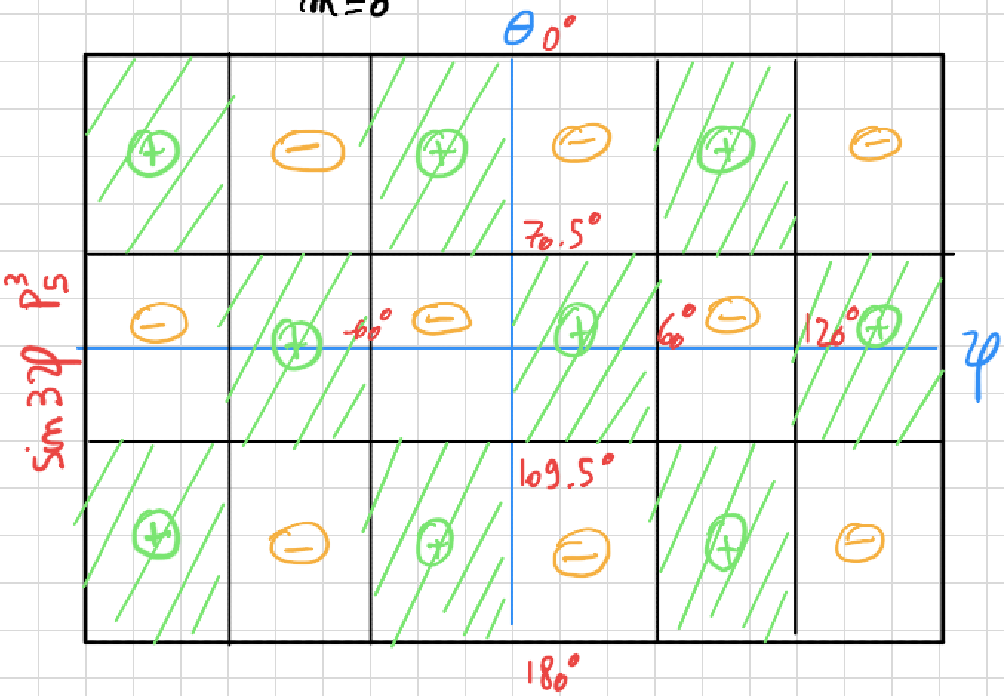
$P_3(z) = \frac{1}{2}(5z^3-3z) = \frac{1}{8}(5\cos 3\theta + 3\cos\theta)$

Combining f_2 and f_3, we get $\cos m\varphi \, P_n^m(\cos\theta)$ and $\sin(m\varphi) P_n^m(\cos\theta)$ which are periodic on the surface of the sphere.

They have $n-m$ nodal lines parallel to the equator and m nodal lines along lines of longitude (roots of $P_n^m(\cos\theta)$ and of $\cos m\varphi$ or $\sin m\varphi$, respectively). This divides the sphere into rectangular domains, or tesserae. $\frac{\cos}{\sin} m\varphi \, P_n^m(\cos\theta)$ are called even/odd tesseral harmonics.

$\Rightarrow \exists \; 2n+1$ tesseral harmonics of degree n. Summing them defines the spherical surface harmonics of degree n:

$$Y_n(\theta,\varphi) = \sum_{m=0}^{n} \left(a_{nm} \cos m\varphi + b_{nm} \sin m\varphi\right) P_n^m(\cos\theta) \quad (7)$$



From $P_n^m(\eta) = \dfrac{(1-\eta^2)^{m/2}}{2^n\, n!} \dfrac{d^{n+m}(\eta^2-1)^n}{d\eta^{n+m}}$, integrating by part we can prove that when $n \neq l$ or $m \neq l$

$$\text{(8)} \begin{cases} \displaystyle\int_{-1}^{1} P_n^m(\eta)\, P_l^m(\eta)\, d\eta = 0 \qquad \int_{-1}^{1} P_n^m(\eta)\, P_n^l(\eta)\, \frac{d\eta}{1-\eta^2} = 0 \quad (m \neq l) \\ \displaystyle\int_{-1}^{1} \left[P_n^m(\eta)\right]^2 d\eta = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \\ \displaystyle\int_{-1}^{1} \left[P_n^m(\eta)\right]^2 \frac{d\eta}{1-\eta^2} = \frac{1}{m} \frac{(n+m)!}{(n-m)!} \end{cases}$$

From there, follows the expansion theorem. Let $g(\theta, \varphi)$ be a C^2 function on the sphere. Then:

$$\boxed{\begin{aligned} g(\theta,\varphi) &= \sum_{n=0}^{+\infty} a_{n0} P_n(\cos\theta) + \sum_{m=1}^{n} \left(a_{nm} \cos m\varphi + b_{nm} \sin m\varphi\right) P_n^m(\cos\theta) \\ a_{n0} &= \frac{2n+1}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\; g(\theta,\varphi) P_n(\cos\theta) \sin\theta \, d\theta \, d\varphi \\ a_{nm} &= \frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\; g(\theta,\varphi) P_n^m(\cos\theta) \cos m\varphi \sin\theta \\ b_{nm} &= \frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\; g(\theta,\varphi) P_n^m(\cos\theta) \sin m\varphi \sin\theta \end{aligned}} \quad (9)$$

4. The radial function

We write $f_1 = (kr)^{-1/2} v(r)$ and plug in (1). We get:

$$r^2 \frac{d^2 v}{dr^2} + r \frac{dv}{dr} + \left[k^2 r^2 - \left(m + \frac{1}{2}\right)^2\right] v = 0$$

Changing the variable to $\rho = kr$ we get

$$\frac{d^2 v}{d\rho^2} + \frac{1}{\rho}\frac{dv}{d\rho} + \left(1 - \frac{(m+1/2)^2}{\rho^2}\right) v = 0,$$

a Bessel equation of half order $p = m + \frac{1}{2}$. The solution is thus:

$$\boxed{f_1(r) = \frac{1}{\sqrt{kr}} Z_{m+\frac{1}{2}}(kr)}$$

where $Z_{m+\frac{1}{2}}$ is a cylindrical Bessel function of half order:

- $Z = J$ when the domain includes the origin (first kind)
- $Z = N$, almost never used (blows up at $r=0$)
- $Z = H^{(1)} = J - jN$ for outward radiation only
  $= H^{(2)}$, almost never used, incident spherical radiation

Following notations initially proposed by Morse (1936) we define the spherical Bessel functions as:

$$\boxed{f_1(\rho) = \begin{cases} z_m(\rho) = \sqrt{\frac{\pi}{2\rho}}\, Z_{m+1}(\rho) \\ n_m(\rho) = \sqrt{\frac{\pi}{2\rho}}\, N_{m+\frac{1}{2}}(\rho) \\ h_m^{(2)}(\rho) = \sqrt{\frac{\pi}{2\rho}}\, H^{(2)}_{m+\frac{1}{2}}(\rho) \end{cases} \qquad \begin{aligned} j_m(\rho) &= \sqrt{\frac{\pi}{2\rho}}\, J_{m+\frac{1}{2}}(\rho) \\ h_m^{(1)}(\rho) &= \sqrt{\frac{\pi}{2\rho}}\, H^{(1)}_{m+\frac{1}{2}}(\rho) \end{aligned}}$$

We'll note the following asymptotic behaviors: $[\rho \to +\infty]$

$$\begin{cases} j_n(\rho) \simeq \dfrac{1}{\rho} \cos\left(\rho - \dfrac{n+1}{2}\pi\right) & n_n(\rho) \simeq \dfrac{1}{\rho} \sin\left(\rho - \dfrac{n+1}{2}\pi\right) \\ h_n^{(1)}(\rho) \simeq \dfrac{1}{\rho} j^{n+1} e^{-j\rho} & h_n^{(2)}(\rho) \simeq \dfrac{1}{\rho} (-j)^{n+1} e^{j\rho} \end{cases}$$

and the series expansions (useful when ρ is small)

$$\begin{cases} j_n(\rho) = 2^n \rho^n \displaystyle\sum_{m=0}^{+\infty} \frac{(-1)^m (n+m)!}{m!\,(2n+2m+1)!} \rho^{2m} \\ n_n(\rho) = -\dfrac{1}{2^n \rho^{n+1}} \displaystyle\sum_{m=0}^{+\infty} \frac{\Gamma(2n-2m+1)}{m!\,\Gamma(n-m+1)} \rho^{2m} \end{cases}$$

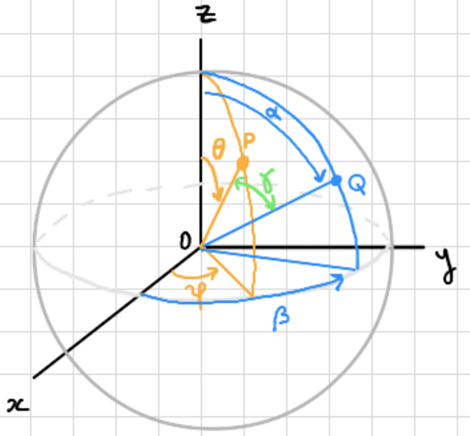
For $n=0$

$$j_0(\rho) = \frac{\sin\rho}{\rho} \qquad n_0(\rho) = \frac{\cos\rho}{\rho} \qquad h_0^{(1)}(\rho) = \frac{e^{-j\rho}}{\rho} \qquad h_0^{(2)} = \frac{e^{j\rho}}{\rho}$$

Summarizing everything, the solution ψ is: (10)

$$\boxed{\psi(r,\theta,\varphi) = \sum_{n=0}^{\infty} z_n(kr)\left[a_{n0} P_n(\cos\theta) + \sum_{m=1}^{n} \left(a_{nm} \cos m\varphi + b_{nm} \sin m\varphi\right) P_n^m(\cos\theta)\right]}$$

5. Addition theorem for the Legendre polynomials



Problem: To express a zonal harmonic $P_n(\eta)$ in terms of a new axis of reference.

Let $P(\theta, \varphi)$ a point and $Q(\alpha, \beta)$ another, referenced in the spherical system defined from $(x\,y\,z)$ by the angles (θ, φ) and (α, β).

The angle between the axes OP and OQ is noted γ. $\cos\gamma$ is the projection of OP on OQ:

$$\vec{OP} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}, \quad OQ = \begin{pmatrix} \sin\alpha\cos\beta \\ \sin\alpha\sin\beta \\ \cos\alpha \end{pmatrix} \Rightarrow \cos\gamma = s\theta\, s\alpha \overbrace{(c\varphi\, c\beta + s\varphi\, s\beta)}^{\cos(\varphi-\beta)} + c\alpha\, c\theta$$

The zonal harmonic at P referred to the new axis OQ is $P_n(\cos\gamma)$. We want to expand $P_n\cos(\gamma)$ as a function of $(\theta, \varphi), \alpha, \beta$.

We assume that $P_n\cos(\gamma)$ expands only into associated Legendre polynomials of order n:

$$P_n(\cos\gamma) = \frac{c_0}{2} P_n(\cos\theta) + \sum_{m=1}^{n} (c_m \cos m\varphi + d_m \sin m\varphi)\, P_n^m(\cos\theta)$$

To find the coefficients c_m, multiply by $P_n^m(\cos\theta)\cos m\varphi$ and integrate:

$$\int_0^{\pi}\int_0^{2\pi} P_n(\cos\gamma)\, P_n^m(\cos\theta)\cos m\varphi \sin\theta\, d\theta\, d\varphi = \text{(orthogonality)}$$

$$= \int_0^{\pi}\int_0^{2\pi} c_m \left[P_n^m(\cos\theta)\right]^2 \cos(m\varphi)^2 s\theta\, d\theta\, d\varphi = c_m \int_{-1}^{1} P_n^m(\eta)^2 d\eta \int_0^{2\pi} \cos^2 m\varphi\, d\varphi$$

Since $\int_{-1}^{1} P_n^m(\eta)^2 d\eta = \frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}$ and $\cos^2 m\varphi = \frac{1}{2}(1 + \cos 2m\varphi)$

$$\int_0^{2\pi} \cos^2 m\varphi \, d\varphi = \pi \quad \Rightarrow$$

$$(a) = \int_0^{2\pi}\int_{-1}^{1} P_n(\cos\gamma)\, P_n^m(\cos\theta)\cos m\varphi \sin\theta \, d\theta\, d\varphi = \frac{2\pi}{2n+1} \frac{(n+m)!}{(n-m)!} c_m$$

For d_m, we do the same with $P_n^m(\cos\theta)\sin m\varphi$. Since $\sin^2 m\varphi = \frac{1}{2}(1-\cos 2m\varphi)$ the calculation is the same:

$$(b) = \int_0^{2\pi}\int_{-1}^{1} P_n(\cos\gamma)\, P_n^m(\cos\theta)\sin m\varphi \sin\theta \, d\theta\, d\varphi = \frac{2\pi}{2n+1} \frac{(n+m)!}{n-m!} d_m$$

To go further, we use the fact that for any function $g(\theta,\varphi)$ that satisfy the condition of the expansion theorem (9), the value at the pole $\theta=0$ is

$$\left[g(\theta,\varphi)\right]_{\theta=0} = \sum_{n'=0}^{\infty} a_{n'0} = \frac{1}{4\pi}\sum_{n'=0}^{\infty}(2n'+1)\int_0^{2\pi}\int_0^{\pi} g(\theta,\varphi)\, P_{n'}(\cos\theta)\sin\theta\, d\theta\, d\varphi$$

In particular, for any surface harmonic $g(\theta,\varphi) = Y_n(\theta,\varphi)$, by orthogonality one term remains in the sum, and we get:

$$\int_0^{2\pi}\int_0^{\pi} Y_n(\theta,\varphi)\, P_n(\cos\theta)\sin\theta\, d\theta\, d\varphi = \frac{4\pi}{2n+1}\left[Y_n(\theta,\varphi)\right]_{\theta=0}$$ ← on the pole of $P_n(\cos\theta)$

This is precisely the kind of integral in (a) and (b)

$$(a) = \frac{4\pi}{2n+1}\left[P_n^m(\cos\theta)\cos m\varphi\right]_{\gamma=0} = P_n^m(\cos\alpha)\cos(m\beta)$$

since at $\gamma=0$, $\theta=\alpha$ and $\varphi=\beta$

$$(b) = \left[P_n^m(\cos\theta) \sin m\varphi \right]_{\gamma=0} = P_n^m(\cos\alpha) \sin m\beta$$

$$\Rightarrow \begin{cases} c_m = 2 \dfrac{(n-m)!}{(n+m)!} P_n^m(\cos\alpha) \cos m\beta \\ d_m = 2 \dfrac{(n-m)!}{(n+m)!} P_n^m(\cos\alpha) \sin m\beta \end{cases}$$

And we get the addition formula for Legendre polynomials: (11)

$$\boxed{\begin{aligned} &P_n(\cos\gamma) = P_n(\cos\alpha) P_n(\cos\theta) + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\alpha) P_n^m(\cos\theta) \cos[m(\varphi - \beta)] \\ &\text{with } \cos\gamma = \sin\theta \sin\alpha \cos(\varphi - \beta) + \cos\theta \cos\alpha \end{aligned}}$$

We will use it to expand plane waves traveling in a direction (α, β) not aligned with the z axis.

6. Expansion of plane waves

We now expand a plane wave as a sum of spherical waves. Let's assume the propagation vector k is in the (α, β) direction:

$$k_x = k \sin\alpha \cos\beta \qquad k_y = k \sin\alpha \sin\beta \qquad k_z = k \cos\alpha$$

while the coordinates of any observation point are:

$$x = r \sin\theta \cos\varphi \qquad y = r \sin\theta \sin\varphi \qquad z = r \cos\theta$$

The phase is then:

$$\vec{k}.\vec{r} = kr\left[\sin\alpha \sin\theta \cos(\varphi - \beta) + \cos\alpha \cos\theta\right] = kr \cos\gamma$$

The plane wave $f = e^{-jkr\cos\gamma}$ is a solution of $\nabla^2\psi + k^2\psi = 0$ that is C^2 everywhere including at the origin. We can expand it using (10). Considering for now an axis in the same direction as the wave, the wave must be symmetrical about the axis and:

$$\boxed{e^{-jkr\cos\gamma} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\gamma)}$$

Multiply by $P_n(\cos\gamma)\sin\gamma$ and integrate from $\gamma = 0$ to π:

$$a_n j_n(kr) = \frac{2n+1}{2} \int_0^{\pi} e^{-jkr\cos\gamma} P_n(\cos\gamma) \sin\gamma \, d\gamma$$

Trick: We get rid of the dependency on r by differentiating n times with respect to $\rho = kr$, then take $\rho = 0$. It works because $\left[\frac{d^n j_n(\rho)}{d\rho^n}\right]_{\rho=0} = \frac{2^n n!^2}{(2n+1)!}$, which can easily be proven from the series expansion of $j_n(\rho)$.

$$\frac{2^n (n!)^2}{(2n+1)!} a_n = \frac{2n+1}{2} (-j)^n \int_0^{\pi} \cos^n\gamma \, P_n(\cos\gamma) \sin\gamma \, d\gamma$$

$$= \frac{2n+1}{2} (-j)^n \int_{-1}^{1} \eta^n P_n(\eta) \, d\eta$$

$$= \frac{2n+1}{2} (-j)^n \frac{1}{2^n n!} \int_{-1}^{1} \eta^n \frac{d^n}{d\eta^n} (\eta^2-1)^n = \frac{2n+1}{2} (-j)^n \frac{1}{2^n n!} I_n$$

$$\Rightarrow \quad a_n = \left(\frac{2n+1}{2}\right) (-j)^n \underbrace{\frac{(2n+1)!}{4^n (n!)^3} I_n}_{= 2 \text{ (Mathematica)}} \Rightarrow \quad a_n = (2n+1)(-j)^n$$

$$\Rightarrow \quad \boxed{e^{-jkr\cos\gamma} = \sum_{n=0}^{\infty} (-j)^n (2n+1) \, j_n(kr) \, P_n(\cos\gamma)}$$

If the z axis is aligned with the plane wave direction, use this formula with $\gamma = \theta$. Otherwise, use addition theorem:

$$\boxed{\begin{aligned} e^{-jkr\cos\gamma} = \sum_{n=0}^{+\infty} (-j)^n (2n+1) \, j_n(kr) \Big\{ & P_n(\cos\alpha) P_n(\cos\theta) + \\ & 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\alpha) \, P_n^m(\cos\theta) \cos m(\varphi - \beta) \Big\} \end{aligned}}$$

7. Addition theorem for the radial function

Let $P(r_0, \theta, \varphi)$ be a point of observation and $Q(r_1, \theta_1, \varphi_1)$ the source point of a spherical wave. The distance QP is:

$$\boxed{\begin{aligned} QP = r &= \sqrt{r_0^2 + r_1^2 - 2 r_0 r_1 \cos\gamma} \\ \cos\gamma &= \sin\theta \sin\theta_1 \cos(\varphi - \varphi_1) + \cos\theta \cos\theta_1 \end{aligned}}$$

One can prove the following addition theorem:

$$\boxed{\begin{aligned} j_0(kr) &= \frac{\sin kr}{kr} = \sum_{m=0}^{\infty} (2m+1)\, P_m(\cos\gamma)\, j_m(kr_0)\, j_m(kr_1) \\ h_0^{(1)}(kr) &= \frac{e^{-jkr}}{-jkr} = \begin{cases} \sum_{m=0}^{\infty} (2m+1)\, P_m(\cos\gamma)\, j_m(kr_0)\, h_m^{(1)}(kr_1) & r_1 > r_0 \\ \sum_{m=0}^{\infty} (2m+1)\, P_m(\cos\gamma)\, j_m(kr_1)\, h_m^{(1)}(kr_0) & r_0 > r_1 \end{cases} \end{aligned}}$$

This is proven as a particular case of the so-called Gegenbauer addition theorem in [1] and with a simpler Green's function approach in [2]

1. A treatise on the theory of Bessel functions, George Neville Watson
2. Mathematics of Classical and Quantum Physics, Frederick W. Byron and Robert W. Fuller.